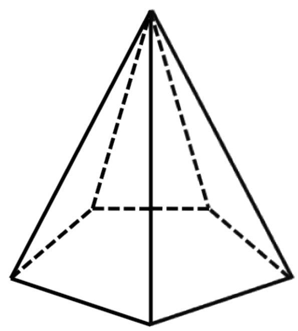
Exam Symmetry in Physics

Date	April 6, 2020
Time	8:30 - 11:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- All subquestions (a, b, etc) of the three exercises have equal weight
- Illegible answers will be not be graded
- Good luck!

Exercise 1

Consider a regular five-sided pyramid with a regular pentagon as base (see figure) and its symmetry group C_{5v}.



(a) Identify all transformations that leave this regular five-sided pyramid invariant.

(b) Show that C_{5v} is isomorphic to D_5, for instance using cycle notation.

(c) Argue, using geometrical arguments, that C_{5v} has four conjugacy classes.

(d) Determine the dimensions of all inequivalent irreps of C_{5v}.

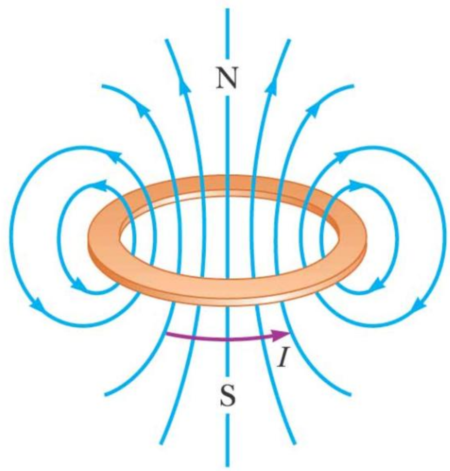
(e) Construct explicitly the three-dimensional vector representation D^V for the two transformations that generate C_{5v} and extract a two-dimensional irrep from it.

(f) Construct the character table of C_{5v}. The irrep obtained in (e) may be used and it may be convenient to use $x \equiv \cos(2\pi/5) = -\frac{1}{4}(1-\sqrt{5})$ and $y \equiv \cos(4\pi/5) = -\frac{1}{4}(1+\sqrt{5})$, that satisfy $x^2+y^2=\frac{3}{4}$ and $xy=-\frac{1}{4}$.

(g) Decompose D^V of C_{5v} into irreps and use this to conclude whether this group allows in principle for an invariant three-dimensional vector, such as an electric dipole moment.

Exercise 2

Consider a circular electric current loop and the magnetic field it generates, as displayed in the figure:



(a) Identify all symmetry transformations that leave this system invariant and call the group that they form G_{loop}.

(b) Explain why G_{loop} cannot be isomorphic to the group $O(2)$ of orthogonal 2×2 matrices.

(c) Explain why G_{loop} cannot be isomorphic to the group $U(1)$ of unitary 1×1 matrices.

(d) Give a nontrivial one-dimensional complex irreducible representation of the group $SO(2)$ of orthogonal 2×2 matrices with determinant equal to 1, and argue whether that is also an irreducible representation of G_{loop}.

Exercise 3

Consider the group $SO(3)$ of orthogonal 3×3 matrices with determinant equal to 1. Consider its action on the angular momentum states $|l, m\rangle$ through the operator

$$O(\theta, \hat{n}) = \exp\left(\frac{i}{\hbar}\theta\, \hat{n} \cdot \vec{L}\right).$$

(a) Write down the explicit matrix for L_z acting on the space of $|1, m\rangle$ states.

(b) Write down the explicit matrix for $O(\theta, \hat{n})$ acting on the space of $|1, m\rangle$ states for the specific case $\hat{n} = \hat{z}$. Call this matrix $D^{(l=1)}(\theta)$ and determine the range of θ.

(c) Use the character of this $l = 1$ representation $D^{(l=1)}$ to show that it is equivalent to the vector representation D^V of $SO(3)$.

(d) Show that $D^{(l=1)} \notin SO(3)$ and explain why it can nevertheless be a representation of $SO(3)$.